5, i.e., $r = 5$ and the number of scaling factors is 8, i.e., $\beta = 8$. Thus, the minimum number of measured modes to have an overspecified system of linear equations is two. The measured modes are simulated by calculating the natural frequencies and the mode shapes at the test DOF using the exact mass and stiffness matrices in the finite element DOF. Three modes (modes 1, 2, and 3) are employed for the correction. The identified scaling factors are $\{s\} = \{10.20 \quad -11.09 \quad 20.43 \quad 0.27 \quad -15.62 \quad -4.56 \quad -0.035 \quad 0.040\}^T$.

The identified scaling factors can be employed to correct the mass and stiffness matrices either in the finite element DOF using Eqs. (2) and (3) or in the test DOF using Eq. (7). The corrected stiffness and mass matrix coefficients in the finite element DOF are shown in Table 1. The mass and stiffness matrix coefficients are not exactly corrected because the measured modes in the test DOF are used for the correction and the reduced model is not exact.

Concluding Remarks

A systematic approach to improve an analytical model of a linear structure in order to match the experimentally identified mode shapes and frequencies was presented. The incorporation of the submatrix concept enables the method to preserve the connectivity and consistency of the stiffness matrix. Orthogonalization of the measured mode shapes with respect to the mass matrix is not required in the method. In order to perform a direct test/analysis correlation, the Guyan reduction technique was employed in the development of the method. The accuracy of the adjusted mass and stiffness matrices using a test analyses model depends not only on the accuracy of the measured modes but also on the accuracy of the TAM. Accurate TAM may be required to perform analytical model improvement properly. A computationally efficient pseudoinverse solution is used to solve the resulting system of linear equations for the scaling factors. If only the mass or the stiffness matrix is incorrect, the adjustment of the incorrect matrix is possible by setting the appropriate submatrix scaling factors to zero.

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Improved Approximate Methods for Computing Eigenvector Derivatives in Structural Dynamics

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Introduction

METHODS of computing eigenvector derivatives have been an active area of research since the earlier work of Fox and Kapoor.[1] Recent development in this area was surveyed by Haftka and Adleman.[2] To obtain exact solutions, Nelson's method[3] appears to be more efficient than other approaches. An improvement to the truncated modal summation representation of eigenvector derivatives was presented in a work by Wang,[4] in which a mode-acceleration type approach was used to obtain a static solution to approximate the contribution due to unavailable higher modes. The numerical performance of this method was compared with several other methods by Sutter et al.[5]

In this Note two improved approximate methods are represented. The first method is the explicit method (EM), which is the method presented in Ref. 4. Based on the same data used by the explicit method, a new implicit method (IM) is developed in this Note. In the implicit method the eigenvector derivatives are assumed to be spanned by the truncated mode shapes together with a residual static mode. The unknown coefficients are computed by a Bubnov-Galerkin method from the governing equation for eigenvector derivatives. Numerical examples show that this method drastically improves solution accuracy.

Analysis

The eigenvalue problem for undamped systems in structural dynamics is

$$K\phi = \lambda M\phi \tag{1}$$

Taking partial derivatives of Eq. (1) with respect to a design variable x yields the following governing equation for eigenvector derivatives:

$$Z_l \frac{\partial \phi_l}{\partial x} = F_l \tag{2}$$

where

$$Z_l = K - \lambda_l M \tag{3}$$

$$F_l = -\frac{\partial Z_l}{\partial x}\phi_l \tag{4}$$

eigenvalue:

$$\lambda_l = l\text{th}$$

eigenvector:

$$\phi_l = l\text{th}$$

and

$$\frac{\partial Z_l}{\partial x} = \frac{\partial K}{\partial x} - \frac{\partial \lambda_l}{\partial x} M - \lambda_l \frac{\partial M}{\partial x} \tag{5}$$

The eigenvalue derivative is given by

$$\frac{\partial \lambda_l}{\partial x} = \phi_l^T \left(\frac{\partial K}{\partial x} - \lambda_l \frac{\partial M}{\partial x}\right)\phi_l \tag{6}$$

where, for convenience, orthonormal modes are used. That is,

$$\phi_l^T M \phi_l = 1 \tag{7}$$

Exact Solution of Eigenvector Derivatives

Since the matrix Z_l is singular, the eigenvector derivatives cannot be obtained from Eq. (2) by elementary means. However, several methods for computing eigenvector derivatives have been developed in the literature. They are all derived based on Eqs. (2) and (7). For an N-degree-of-freedom system, if all the modes are available, the eigenvector derivative can be computed using the following equations[1]:

$$\frac{\partial \phi_l}{\partial x} = c_l\,\phi_l + \sum_{\substack{i=1\\ i\neq l}}^{N} c_i\,\phi_i \tag{8}$$

where

$$c_l = -\frac{1}{2}\phi_l^T \frac{\partial M}{\partial x}\phi_l \tag{9}$$

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$$c_i = -\frac{\phi_i^T \frac{\partial Z_l}{\partial x} \phi_l}{\lambda_i - \lambda_l} \tag{10}$$

Algebraic methods of computing eigenvector derivatives using only the eigenvector of concern were also available in the literature[1,3] These methods require special manipulation of the system matrices and cannot be implemented easily. It should be noted that other algebraic methods such as singular value decomposition[6] are also available to solve Eq. (2).

Approximate Solution of Eigenvector Derivative

For large structural systems usually only a truncated set of mode shapes are computed. Thus, as in the case of forced dynamic response analysis that uses the normal mode method, an approximate solution can be obtained by use of the modal summation method using only the available modes.

Improved Approximate Methods

Explicit Method

Assume that $\hat{N}$ modes are computed for a system with N degrees of freedom. The exact eigenvector derivative [Eq. (8)], can be written as

$$\frac{\partial \phi_l}{\partial x} = \sum_{i=1}^{\hat{N}} c_i \, \phi_i + S_R \tag{11}$$

where

$$S_R = \sum_{j=\hat{N}+1}^{N} c_j \, \phi_j \tag{12}$$

In the explicit method we seek an approximation of the contribution due to higher modes (S_R) without using higher-mode data. Using the results of Eqs. (8–10), Eq. (12) can be written as

$$S_R = \sum_{j=\hat{N}+1}^{N} \frac{\phi_j^T F_l}{\lambda_l - \lambda_j} \phi_j \tag{13}$$

When $l << \hat{N}$, $\lambda_l << \lambda_j$, Eq. (13) can be approximated by

$$S_{RA} = \sum_{j=\hat{N}+1}^{N} \frac{\phi_j^T F_l}{\lambda_j} \phi_j \tag{14}$$

or

$$S_{RA} = \left(\sum_{j=1}^{N} \frac{\phi_j^T F_l \, \phi_j}{\lambda_j} - \sum_{j=1}^{\hat{N}} \frac{\phi_j^T F_l \phi_j}{\lambda_j} \right) \tag{15}$$

The first terms of Eq. (15) can be shown to be

$$\sum_{j=1}^{N} \frac{\phi_j^T F_l \, \phi_j}{\lambda_j} = K^{-1} F_l \tag{16}$$

Define a static mode as

$$y_l = K^{-1} F_l \tag{17}$$

Also define

$$h_l = \sum_{j=1}^{\hat{N}} \frac{\phi_j^T F_l}{\lambda_j} \phi_j \tag{18}$$

Using these definitions, Eq. (15) becomes

$$S_{RA} = y_l - h_l \tag{19}$$

By substituting S_{RA} for S_R in Eq. (11), we obtain the explicit improved eigenvector derivatives:

$$\frac{\partial \phi_l}{\partial x} \approx \sum_{i=1}^{\hat{N}} c_i \, \phi_i + w_l \tag{20}$$

where

$$w_l = y_l - h_l \tag{21}$$

and y_l and h_l are given by Eqs. (17) and (18), respectively. Since w_l is the static mode y_l with the contribution from lower modes filtered out, hereafter it will be called the residual static mode.

Implicit Method

From Eq. (20) it can be seen that the eigenvector derivative is a linear combination of the lower modes and the residual static mode. Instead of using the explicit form of Eq. (20), we can alternatively assume that

$$\frac{\partial \phi_l}{\partial x} = T_q + c_l \, \phi_l \tag{22}$$

where

$$T = \left[\tilde{\phi} \, \vdots \, w_l \right] \tag{23}$$

and

$$\left[\tilde{\phi} \right] = [\phi_1 \ldots \phi_{\hat{N}}] \tag{24}$$

with ϕ_l excluded.
The unknown coefficients q can be computed via

$$Aq = b \tag{25}$$

Where

$$A = T^T (K - \lambda_l M) T \tag{26}$$

$$b = T^T F_l \tag{27}$$

Substitution of the solution q into Eq. (22) yields

$$\frac{\partial \phi_l}{\partial x} \approx \sum_{i=1}^{\hat{N}} \bar{c}_i \, \phi_i + d_l \, w_l \tag{28}$$

Since the basis vectors in T are M and K orthogonal to each other, the matrix A is diagonal, and it follows that

$$\bar{c}_i = c_i \tag{29}$$

and

$$d_l = \frac{w_l^T F_l}{w_l^T K w_l - \lambda_l \, w_l^T M w_l} \tag{30}$$

Thus, the only difference between the explicit and implicit formulations is in the coefficient of the residual static mode, d_l.

Numerical Example

Consider a four-degree-of-freedom system with mass and stiffness matrices given by

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{kg}$$

and

$$[K] = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 7 & -5 \\ 0 & 0 & -5 & .5 \end{bmatrix} \times 10^5 \text{ N/m}$$

respectively. The derivative of first eigenvector is to be evaluated for the following two cases.

Case 1:

$$\frac{\partial M}{\partial x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \frac{\partial K}{\partial x} = [0]$$

Case 2:

$$\frac{\partial M}{\partial x} = [0], \qquad \frac{\partial K}{\partial x} = 10^5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Table 1 Comparison of eigenvector derivatives

	Exact	Modal superposition (3 modes)	Improved approximation (2 modes plus residue) Explicit	Implicit
		Case 1		
$\frac{\partial \phi_1}{\partial x}$	0.0016	0.0013	0.0016	0.0016
	−0.0006	−0.0004	−0.0006	−0.0006
	−0.0015	−0.0017	−0.0015	−0.0015
	−0.0017	−0.0016	−0.0017	−0.0017
		Case 2		
$\frac{\partial \phi_1}{\partial x}$	0.0085	0.0143	0.0086	0.0085
	0.0245	0.0206	0.0245	0.0245
	−0.0064	−0.0009	−0.0063	−0.0064
	−0.0052	−0.0080	−0.0052	−0.0052

Table 2 Comparison of errors in the approximation of eigenvector derivatives

Truncated modal superposition (3 modes)	Explicit formulation (2 modes plus residue)	Implicit formulation (2 modes plus residue)
	Case 1	
−269E-6	−27.58E-6	−0.0017E-6
180E-6	5.96E-6	−0.3316E-6
255E-6	2.5E-6	−0.7631E-6
132E-6	2.27E-6	−0.4208E-6
	Case 2	
584E-5	4.656E-5	−0.744829E-5
−389E-5	−5.657E-5	0.132666E-5
551E-5	10.264E-5	0.132980E-5
285E-5	−5.5097E-5	−0.097256E-5

Table 3 Comparison modal components of exact and approximate eigenvector derivatives

	Exact	Modal superposition (3 modes)	Improved approximation (2 modes plus residue) Explicit	Implicit
		Case 1		
c_1	−0.410674E-2	−0.410674E-2	−0.410674E-2	−0.410674E-2
c_2	−0.150286E-2	−0.150286E-2	−0.150286E-2	−0.150286E-2
c_3	−0.146456E-2	−0.146456E-2	−0.143687E-2	−0.146382E-2
c_4	0.063354E-2	0	0.062331E-2	0.063600E-2
		Case 2		
c_1	0	0	0	0
c_2	−0.0361429	−0.0361429	−0.0361429	−0.0361429
c_3	−0.0030144	−0.0030144	−0.0028574	−0.0030063
c_4	−0.0137192	0	−0.0134977	−0.0137207

The derivatives of the first eigenvector for the two cases are summarized in Table 1. These results show that, when two modes plus the residual mode are used, both the explicit and implicit formulations yielded much better results than the truncated modal solution using three modes. These results were computed by MATLAB.[7] Table 2 summarizes the errors for various approximate methods. It shows that the errors in the implicit method were an order of magnitude smaller than those in the explicit formulation.

In order to explain why the improved methods are effective, the eigenvector derivatives are represented by modal expansion of the eigenvectors in the following form:

$$\frac{\partial \phi_1}{\partial x} = \sum_{i=1}^{4} c_i \, \phi_i \tag{31}$$

The coefficients c_i for various solutions for $\partial\phi_1/\partial x$ are summarized in Table 3. From these results it is clear that the implicit method performs best due to its ability to include the contribution of truncated modes more exactly than the explicit method. For the truncated modal solution, since the contribution of mode 4 is ignored, a large error may result, as in case 2, due to the significant contribution of mode 4.

Concluding Remarks

Two improved approximate methods for computing eigenvector derivatives in structural dynamics are presented in this paper. The methods are an improvement to truncated modal superposition of eigenvector derivatives using the static solution to account for the contribution of truncated higher modes approximately. The effectiveness of the explicit formulation was demonstrated in Ref. 5. Since the implicit method is based on a Bubnov-Galerkin procedure, it is the optimum solution for the eigenvector derivatives when the residual static mode is introduced into the basis vector. Thus, whenever the residual static mode is available, the implicit method can be used to further enhance the solution accuracy with little extra computational cost.

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